

# Blaschke Isoparametric Hypersurfaces in the Conformal Space $\mathbb{Q}_1^{n+1}$ , II

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## Abstract

Let  $x : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$  be a regular space-like hypersurface in the conformal space  $\mathbb{Q}_1^{n+1}$ . We classify all those hypersurfaces with parallel Blaschke tensor in the conformal space up to the conformal equivalence.

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## § 1. Introduction.

In [7] we have classified all regular space-like hypersurfaces with two distinct constant Blaschke eigenvalues in the conformal space  $\mathbb{Q}_1^{n+1}$ . In this paper, we shall focus on a special class of Blaschke isoparametric hypersurfaces, which are called hypersurfaces with parallel Blaschke tensor. We shall find that those hypersurfaces with parallel Blaschke tensor must be of 1, 2, or 3 distinct constant Blaschke eigenvalues.

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First of all, we shall present some concrete space-like Blaschke isoparametric hypersurfaces in  $\mathbb{Q}_1^{n+1}$  with two distinct Blaschke eigenvalues. The details of calculation will occur in the proof of the main theorem of this paper later.

**Example 1.1.** For some real number  $r > 0$  and integers  $n, k$  satisfying  $k = 1, \dots, n-1$ ,  $u : \mathbf{N}^k \rightarrow \mathbb{S}_1^{k+1}(r) \subset \mathbb{R}_1^{k+2}$  be a regular maximal space-like hypersurface with constant scalar curvature

$$\rho_1 = \frac{k(k-1)}{r^2} - \frac{n-1}{n}.$$

Let  $\{e_1, \dots, e_k\}$  be an local basis for  $u$  with dual basis  $\{\omega^1, \dots, \omega^k\}$ . Denote the second fundamental form of  $u$  by  $II_1 = \sum_{ij} h_{ij} \omega^i \otimes \omega^j$ . Then

$$\sum_i h_i^i = 0, \quad \sum_{ij} h_j^i h_i^j = \frac{n-1}{n},$$

where the indices are shifted by the first fundamental form  $I_1$ . Denote  $\Delta_1$  the Laplacian with respect to  $I_1$ . It is easy to know that

$$\Delta_1 u = -\frac{ku}{r^2}.$$

Let  $v : \mathbb{H}^{n-k}(r) \rightarrow \mathbb{R}_1^{n-k+1}$  be the standard totally umbilical hypersurface. Then the scalar curvature

$$\rho_2 = -\frac{(n-k)(n-k-1)}{r^2}.$$

And we have

$$\Delta_2 v = \frac{(n-k)v}{r^2},$$

where  $\Delta_2$  is the Laplacian with respect to the first fundamental form  $I_2$  of  $v$ . For  $y = (u, v) : \mathbf{M} = \mathbf{N}^k \times \mathbb{H}^{n-k}(r) \rightarrow C^{n+2} \subset \mathbb{R}_2^{n+3}$ , we find exactly that the conformal metric  $g = \langle dy, dy \rangle$ . Therefore  $y$  is the canonical lift of  $x = [y] : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$ . A direct calculation will yield that

$$(A_j^i) = \frac{1}{2r^2}(\mathbf{I}_k \oplus (-\mathbf{I}_{n-k})), \quad (B_j^i) = (h_j^i) \oplus \mathbf{0}_{n-k}, \quad C_i = 0, \forall i,$$

where  $\mathbf{I}_k$  means  $k$ -rank unit matrix and  $\mathbf{0}_k$  means  $k$ -rank zero matrix.

**Example 1.2.** For some real number  $r > 0$  and integers  $n, k$  satisfying  $k = 1, \dots, n-1$ ,  $u : \mathbf{N}^k \rightarrow \mathbb{H}_1^{k+1}(r) \subset \mathbb{R}_2^{k+2}$  be a regular maximal space-like hypersurface with constant scalar curvature

$$\rho_1 = -\frac{k(k-1)}{r^2} - \frac{n-1}{n}.$$

Let  $\{e_1, \dots, e_k\}$  be an local basis for  $u$  with dual basis  $\{\omega^1, \dots, \omega^k\}$ . Denote the second fundamental form of  $u$  by  $II_1 = \sum_{ij} h_{ij} \omega^i \otimes \omega^j$ . Then

$$\sum_i h_i^i = 0, \quad \sum_{ij} h_j^i h_i^j = \frac{n-1}{n}.$$

Denote  $\Delta_1$  the Laplacian for  $I_1$ . It is easy to know that

$$\Delta_1 u = \frac{ku}{r^2}.$$

Let  $v : \mathbb{S}^{n-k}(r) \rightarrow \mathbb{R}^{n-k+1}$  be the standard totally umbilical hypersurface. Then the scalar curvature

$$\rho_2 = \frac{(n-k)(n-k-1)}{r^2}.$$

And we have

$$\Delta_2 v = -\frac{(n-k)v}{r^2},$$

where  $\Delta_2$  is the Laplacian with respect to the first fundamental form  $I_2$  of  $v$ . For  $y = (u, v) : \mathbf{M} = \mathbf{N}^k \times \mathbb{S}^{n-k}(r) \rightarrow C^{n+2} \subset \mathbb{R}_2^{n+3}$ , we find exactly that the conformal metric  $g = \langle dy, dy \rangle$ . Therefore  $y$  is the canonical lift of  $x = [y] : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$ . A direct calculation will yield that

$$(A_j^i) = -\frac{1}{2r^2}(I_k \oplus (-I_{n-k})), \quad (B_j^i) = (h_j^i) \oplus (\mathbf{0}_{n-k}), \quad C_i = 0, \forall i.$$

Now the main theorem in this paper is stated as follows:

**Theorem B** Let  $x : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$  be a regular space-like hypersurface in the conformal space  $\mathbb{Q}_1^{n+1}$ . If the Blaschke tensor  $\mathbb{A}$  of  $x$  is parallel, then one of the following statements holds:

- (1)  $x$  is conformal isotropic and is therefore locally conformally equivalent to:
  - (a) a maximal hypersurface in  $\mathbb{S}_1^{n+1}$  with constant scalar curvature; or
  - (b) a maximal hypersurface in  $\mathbb{R}_1^{n+1}$  with constant scalar curvature; or
  - (c) a maximal hypersurface in  $\mathbb{H}_1^{n+1}$  with constant scalar curvature;
- (2)  $x$  is of parallel conformal second fundamental form  $\mathbb{B}$  and is therefore locally conformally equivalent to:
  - (a) a standard cylinder  $\mathbb{H}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1+r^2})$ ,  $r > 0$ , in  $\mathbb{S}_1^{n+1}$  for some positive integer  $k$  and  $n-k$ ; or
  - (b) a standard cylinder  $\mathbb{H}^k \times \mathbb{R}^{n-k}$  in  $\mathbb{R}_1^{n+1}$  for some positive integer  $k$  and  $n-k$ ; or
  - (c) a standard cylinder  $\mathbb{H}^k(r) \times \mathbb{H}^{n-k}(\sqrt{1-r^2})$ ,  $0 < r < 1$ , in  $\mathbb{H}_1^{n+1}$  for some positive integer  $k$  and  $n-k$ ; or

(d) the wrapped product embedding

$$u : \mathbb{H}^p(r) \times \mathbb{S}^q(\sqrt{r^2 + 1}) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \subset \mathbb{R}_1^{n+2} \rightarrow \mathbb{R}_1^{n+1},$$

$$(u', u'', t, u''') \mapsto (tu', tu'', u'''),$$

where

$$u' \in \mathbb{H}^p(r), u'' \in \mathbb{S}^q(\sqrt{r^2 + 1}), t > 0, u''' \in \mathbb{R}^{n-p-q-1}, r > 0,$$

for some positive integer  $p, q$ , and  $n - p - q - 1$ ;

(3)  $x$  is non-conformal isotropic with a non-parallel conformal second fundamental form B and is locally conformally equivalent to:

- (a) one of the hypersurfaces as indicated in Example 1.1; or
- (b) one of the hypersurfaces as indicated in Example 1.2.

**Remark 1.4.** All the above examples are Blaschke isoparametric. Among them, case (1) is of one eigenvalue of the Blaschke tensor  $\mathbb{A}$ , the first three examples of case (2) and case (3) are of two eigenvalues, and subcase (d) of case (2) is of three eigenvalues.

This paper is organized as follows. In Section 2 we will give some frequently-used equations occurred many times in [8-10] *etc.*, especially in [7]. Readers may find more details in those conferences. In addition, we prove some lemmas which will be used in the next section. In Section 3 we prove the Theorem B.

## § 2. The fundamental equations and some lemmas.

Let  $\mathbb{R}_s^N$  denote pseudo-Euclidean space, which is the real vector space  $\mathbb{R}^N$  with the non-degenerate inner product  $\langle, \rangle$  given by

$$\langle \xi, \eta \rangle = - \sum_{i=1}^s x_i y_i + \sum_{i=s+1}^N x_i y_i, \quad (1.1)$$

where  $\xi = (x_1, \dots, x_N), \eta = (y_1, \dots, y_N) \in \mathbb{R}^N$ .

Let

$$C^{n+1} := \{\xi \in \mathbb{R}_{s+1}^{n+2} | \langle \xi, \xi \rangle = 0, \xi \neq 0\}, \quad (1.2)$$

$$\mathbb{Q}_s^n := \{[\xi] \in \mathbb{R}P^{n+1} | \langle \xi, \xi \rangle = 0\} = C^{n+1} / (\mathbb{R} \setminus \{0\}). \quad (1.3)$$

We call  $C^{n+1}$  the light cone in  $\mathbb{R}_{s+1}^{n+2}$  and  $\mathbb{Q}_s^n$  the conformal space (or projective light cone) in  $\mathbb{R}P^{n+1}$ .

The standard metric  $h$  of the conformal space  $\mathbb{Q}_s^n$  can be obtained through the pseudo-Riemannian submersion

$$\pi : C^{n+1} \rightarrow \mathbb{Q}_s^n, \xi \mapsto [\xi].$$

We can check  $(\mathbb{Q}_s^n, h)$  is a pseudo-Riemannian manifold.

We define the pseudo-Riemannian sphere space  $\mathbb{S}_s^n(r)$  and pseudo-Riemannian hyperbolic space  $\mathbb{H}_s^n(r)$  with radius  $r$  by

$$\mathbb{S}_s^n(r) = \{u \in \mathbb{R}_s^{n+1} | \langle u, u \rangle = r^2\}, \quad \mathbb{H}_s^n(r) = \{u \in \mathbb{R}_{s+1}^{n+1} | \langle u, u \rangle = -r^2\}.$$

When  $r = 1$  we usually omit the radius  $r$ . When  $s = 1$  and  $r = 1$  we call them de Sitter space  $\mathbb{S}_1^n$  and anti-de Sitter space  $\mathbb{H}_1^n$ .

We may assume  $\mathbb{Q}_s^n$  as the common compactification of  $\mathbb{R}_s^n$ ,  $\mathbb{S}_s^n$  and  $\mathbb{H}_s^n$ , and  $\mathbb{R}_s^n$ ,  $\mathbb{S}_s^n$  and  $\mathbb{H}_s^n$  as the subsets of  $\mathbb{Q}_s^n$  when referring to the conformal geometry.

Let  $x : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$  be a regular space-like hypersurface in the conformal space  $\mathbb{Q}_1^{n+1}$ . We skip the standard procedure of the analysis of the conformal geometry of hypersurfaces. For more details, readers may see [8], *etc.*. We have four important conformal invariants, the conformal metric  $g$ , the Blaschke tensor  $\mathbb{A}$ , the conformal second fundamental form  $\mathbb{B}$ , the conformal form  $\phi$ . Then we have some fundamental equations of  $x$  that are used later as follows:

$$A_{ij,k} - A_{ik,j} = B_{ij}C_k - B_{ik}C_j, \quad (2.1)$$

$$B_{ij,k} - B_{ik,j} = g_{ij}C_k - g_{ik}C_j, \quad (2.2)$$

$$C_{i,j} - C_{j,i} = \sum_{kl} g^{kl} (B_{ik}A_{lj} - B_{jk}A_{li}), \quad (2.3)$$

$$R_{ijkl} = (g_{ik}A_{jl} - g_{il}A_{jk}) + (A_{ik}g_{jl} - A_{il}g_{jk}) - (B_{ik}B_{jl} - B_{il}B_{jk}), \quad (2.4)$$

$$\sum_{ij} B_j^i B_i^j = \frac{n-1}{n}, \quad (2.5)$$

$$\sum_i B_i^i = 0. \quad (2.6)$$

For  $u : \mathbf{M} \rightarrow \mathbf{L}^{n+1}(\epsilon)$ , when  $\epsilon = 0, 1, -1$ ,  $\mathbf{L}^{n+1}(\epsilon)$  denotes  $\mathbf{R}_1^{n+1}$ ,  $\mathbf{S}_1^{n+1}$  and  $\mathbf{H}_1^{n+1}$ , respectively. We have

$$e^{2\tau} = \frac{1}{n-1} (n \sum_{ij} h_j^i h_i^j - (\sum_i h_i^i)^2). \quad (2.7)$$

$$A_{ij} = \tau_i \tau_j - H h_{ij} - \tau_{i,j} - \frac{1}{2} (\sum_i \tau^i \tau_i - H^2 - \epsilon) I_{ij}, \quad (2.8)$$

$$B_{ij} = e^\tau (h_{ij} - H I_{ij}), \quad (2.9)$$

$$C_i = e^{-\tau} (H \tau_i - \sum_j h_{ij} \tau^j - H_i). \quad (2.10)$$

We remind readers that if we call a tensor is parallel then usually we regard the first order covariant differential of the tensor vanishes. So the Blaschke tensor  $\mathbb{A}$  of the regular space-like hypersurface  $x$  is defined parallel if and only if  $A_{ij,k} = 0$ , for any  $i, j, k = 1, \dots, n$ .

Next we introduce several lemmas.

**Lemma 2.1.** Let  $x : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$  be a regular space-like hypersurface. If the Blaschke tensor  $\mathbb{A}$  is parallel, then the conformal form  $\phi$  vanishes.

*Proof* Denote  $B = (B_{ij})$ . If we choose an appropriate orthonormal basis  $e_1, \dots, e_n$ , we can write

$$B = (b_1 I_{m_1}) \oplus \dots \oplus (b_s I_{m_s}),$$

where  $b_1, \dots, b_s$  are distinct conformal eigenvalues and  $m_1, \dots, m_s$  are some positive integers satisfying  $\sum_{t=1}^s m_t = n$ .

From now on we adopt the convention on the ranges of indices in this section:

$$1 \leq i, j, k, l \leq n, \quad 1 \leq t, t' \leq s.$$

Denote

$$\Gamma_t = \{i | B_{ii} = b_t\}.$$

From the equation (2.1) we have

$$B_{ij}C_k = B_{ik}C_j. \quad (2.11)$$

By (2.5) and (2.6), there must be at least two distinct  $t, t'$  such that  $b_t, b_{t'} \neq 0$ . Taking some fixed  $i = j \in \Gamma_t$  in (2.11), we shall find that all the  $C_k$ 's are zero except  $C_i$ . On the other hand, taking some fixed  $i' = j' \in \Gamma_{t'}$  in (2.11), we shall find that all the  $C_k$ 's are zero except  $C_{i'}$ . From above we know that all the  $C_k$ 's are zero. That means, the conformal form  $\phi$  vanishes.  $\square$

**Lemma 2.2.** Let  $x : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$  be a regular space-like Blaschke isoparametric hypersurface. If the Blaschke tensor  $\mathbb{A}$  is parallel, then  $x$  is Blaschke isoparametric. And the number of distinct Blaschke eigenvalues is no greater than 3.

*Proof* Denote  $A = (A_{ij})$ . If we choose an appropriate orthonormal basis  $e_1, \dots, e_n$ , we can write

$$A = (a_1 I_{m_1}) \oplus \dots \oplus (a_s I_{m_s}),$$

where  $a_1, \dots, a_s$  are distinct Blaschke eigenvalues and  $m_1, \dots, m_s$  are some positive integers satisfying  $\sum_{t=1}^s m_t = n$ .

From now on we adopt the convention on the ranges of indices in this section:

$$1 \leq i, j, k, l \leq n, \quad 1 \leq t, t', t'' \leq s.$$

Denote

$$\Gamma_t = \{i | A_{ii} = a_t\}.$$

Taking  $i = j \in \Gamma_t$  in

$$\sum_k A_{ij,k} \omega^k = dA_{ij} - \sum_k A_{kj} \omega_i^k - \sum_k A_{ik} \omega_j^k, \quad (2.12)$$

we shall get  $da_t = 0$  because of  $\omega_j^i + \omega_i^j = 0$ , which implies that  $a_t$ 's are constant. Therefore  $x$  is Blaschke isoparametric.

From the Lemma 2.1, we know that the conformal form  $\phi = 0$ . Therefore by (2.3), we have  $[A, B] = 0$ . From the algebraic and geometric technics, we can modify the orthonormal basis  $e_1, \dots, e_n$  such that the matrix  $B$  can be diagonalized into the form  $B = \text{diag}(b_1, \dots, b_n)$ .

Taking  $i \in \Gamma_t, j \in \Gamma_{t'}, t \neq t'$  in (2.12), we get

$$\omega_j^i = 0, \text{ when } i \in \Gamma_t, j \in \Gamma_{t'}, t \neq t'. \quad (2.13)$$

The above equation will yield

$$0 = d\omega_j^i + \sum_k \omega_k^i \wedge \omega_j^k = \Omega_j^i,$$

which implies by (2.4) that

$$R_{ijij} = a_t + a_{t'} - b_i b_j = 0, \text{ when } i \in \Gamma_t, j \in \Gamma_{t'}, t \neq t'. \quad (2.14)$$

Fixing one of two subscripts  $i$  and  $j$  in (2.14) and letting the other subscript vary, we will easily obtain  $B = (b_1 I_{m_1}) \oplus \dots \oplus (b_s I_{m_s})$ . So (2.14) comes to

$$a_t + a_{t'} - b_t b_{t'} = 0, \text{ where } t \neq t'. \quad (2.15)$$

If the number of distinct Blaschke eigenvalues  $s > 3$ , it is easy to induce by (2.15) that, for distinct three number  $t, t', t''$ ,

$$a_t - a_{t'} = b_{t''}(b_t - b_{t'}), \quad (2.16)$$

$$\frac{a_t - a_{t'}}{b_t - b_{t'}} = b_{t''}. \quad (2.17)$$

In fact, we can guarantee that when  $t \neq t'$ ,  $b_t \neq b_{t'}$  by (2.16). From (2.17) we know that for fixed  $t, t'$ , the number  $\frac{a_t - a_{t'}}{b_t - b_{t'}}$  will have more than 2 values. That is a contraction. So the assumption  $s > 3$  is wrong. The number of distinct Blaschke eigenvalues is no greater than 3.  $\square$

**Lemma 2.3.** Let  $x : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$  be a regular space-like Blaschke isoparametric hypersurface with parallel Blaschke tensor. If  $x$  has three distinct Blaschke eigenvalues, then the conformal second fundamental form  $\mathbb{B}$  is parallel.

*Proof* From the proof of the Lemma 2.2, we can choose an appropriate orthonormal basis  $e_1, \dots, e_n$  such that

$$A = (a_1 I_{m_1}) \oplus (a_2 I_{m_2}) \oplus (a_3 I_{m_3}), B = (b_1 I_{m_1}) \oplus (b_2 I_{m_2}) \oplus (b_3 I_{m_3}),$$

where  $a_1, a_2, a_3$  are distinct constant Blaschke eigenvalues.

By (2.16) we know that  $b_1, b_2, b_3$  are distinct and non-zero. Combining (2.5), (2.6) and (2.15), we can compute that  $b_1, b_2, b_3$  are constant.

For  $t = 1, 2, 3$ , taking  $i, j \in \Gamma_t$  in

$$\sum_k B_{ij,k} \omega^k = dB_{ij} - \sum_k B_{kj} \omega_i^k - \sum_k B_{ik} \omega_j^k, \quad (2.18)$$

we shall get

$$B_{ij,k} = 0, \text{ where } i, j \in \Gamma_t, \forall t, k. \quad (2.19)$$

Taking  $i \in \Gamma_1, j \in \Gamma_2$  in (2.18), and recalling (2.13), we obtain

$$B_{ij,k} = 0, \text{ where } i \in \Gamma_1, j \in \Gamma_2, k \in \Gamma_3. \quad (2.20)$$

From Lemma 2.1 and (2.2), we know that  $B_{ij,k}$  is all symmetric with respect to the subscripts. Combining (2.19) and (2.20), we know that  $B_{ij,k} = 0, \forall i, j, k$ .  $\square$

### § 3. Proof of the Theorem B.

Let  $x : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$  be a regular space-like Blaschke isoparametric hypersurface with parallel Blaschke tensor. From the Lemma 2.2, we know that the number of distinct Blaschke eigenvalues is no greater than 3.

It suffices to consider the following three cases:

Case (I): the number of distinct Blaschke eigenvalues is 1.

In this case, we know that  $x$  is actually conformal isotropic. In the Theorem 5.2 of [8], we have classified the conformal isotropic submanifolds in the conformal space  $\mathbb{Q}_p^n$ . So we have the

**Theorem 3.1** Let  $x : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$  be a regular space-like hypersurface in the conformal space  $\mathbb{Q}_1^{n+1}$ . If  $x$  is conformal isotropic, then  $x$  is locally conformally equivalent to:

- (a) a maximal hypersurface in  $\mathbb{S}_1^{n+1}$  with constant scalar curvature; or
- (b) a maximal hypersurface in  $\mathbb{R}_1^{n+1}$  with constant scalar curvature; or



(c) a maximal hypersurface in  $\mathbb{H}_1^{n+1}$  with constant scalar curvature.

Case (II): the number of distinct Blaschke eigenvalues is 2.

In this case, we note that we have classified all the space-like Blaschke isoparametric hypersurfaces with two distinct Blaschke eigenvalues in the Theorem A of [11]. For the matter of completeness we state the following modifying version of the Theorem A. We have the

**Theorem 3.2** Let  $x : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$  be a regular space-like hypersurface in the conformal space  $\mathbb{Q}_1^{n+1}$ . If  $x$  is of two distinct constant Blaschke eigenvalues and of vanishing conformal form, then one of the following statements holds:

(1)  $x$  is conformal isoparametric with two distinct conformal principal curvatures and is therefore locally conformally equivalent to the regular space-like hypersurface in  $\mathbb{Q}_1^{n+1}$  determined by:

(a) a standard cylinder  $\mathbb{H}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1+r^2})$ ,  $r > 0$  in  $\mathbb{S}_1^{n+1}$  for some positive integer  $k$  and  $n-k$ ; or

(b) a standard cylinder  $\mathbb{H}^k \times \mathbb{R}^{n-k}$  in  $\mathbb{R}_1^{n+1}$  for some positive integer  $k$  and  $n-k$ ; or

(c) a standard cylinder  $\mathbb{H}^k(r) \times \mathbb{H}^{n-k}(\sqrt{1-r^2})$ ,  $0 < r < 1$ , in  $\mathbb{H}_1^{n+1}$  for some positive integer  $k$  and  $n-k$ ;

(2)  $x$  is locally conformally equivalent to:

(a) one of the hypersurfaces as indicated in Example 1.1; or

(b) one of the hypersurfaces as indicated in Example 1.2.

Case (III): the number of distinct Blaschke eigenvalues is 3.

In this case, from the lemma 2.3 we know that the conformal second fundamental form  $\mathbb{B}$  is parallel. But we have classified all the space-like hypersurfaces with parallel conformal second fundamental forms in the Classification Theorem of [10]. For the matter of completeness we state the following modifying version of the Classification Theorem. We have the

**Theorem 3.3** Let  $x : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$  be a regular space-like hypersurface in the conformal space  $\mathbb{Q}_1^{n+1}$ . If  $x$  is of parallel conformal second fundamental form  $\mathbb{B}$ , then it is locally conformally equivalent to:

(a) a standard cylinder  $\mathbb{H}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1+r^2})$ ,  $r > 0$ , in  $\mathbb{S}_1^{n+1}$  for some positive integer  $k$  and  $n-k$ ; or

(b) a standard cylinder  $\mathbb{H}^k \times \mathbb{R}^{n-k}$  in  $\mathbb{R}_1^{n+1}$  for some positive integer  $k$  and  $n-k$ ; or

(c) a standard cylinder  $\mathbb{H}^k(r) \times \mathbb{H}^{n-k}(\sqrt{1-r^2})$ ,  $0 < r < 1$ , in  $\mathbb{H}_1^{n+1}$  for some positive integer  $k$  and  $n-k$ ; or

(d) the wrapped product embedding

$$u : \mathbb{H}^p(r) \times \mathbb{S}^q(\sqrt{r^2+1}) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \subset \mathbb{R}_1^{n+2} \rightarrow \mathbb{R}_1^{n+1},$$

$$(u', u'', t, u''') \mapsto (tu', tu'', u'''),$$

where

$$u' \in \mathbb{H}^p(r), u'' \in \mathbb{S}^q(\sqrt{r^2 + 1}), t > 0, u''' \in \mathbb{R}^{n-p-q-1}, r > 0,$$

for some positive integer  $p, q$ , and  $n - p - q - 1$ .

The last thing we ought to prove is that the wrapped product embedding  $u$  has 3 distinct constant Blaschke eigenvalues. In fact, the first fundamental form of  $u$  is

$$I = \langle du, du \rangle = t^2 du' \cdot du' + t^2 \langle du'', du'' \rangle + dt \otimes dt + du''' \cdot du'''.$$

The unit time-like normal vector field of  $u$

$$e_{n+1} = \left( \frac{r}{\sqrt{r^2 + 1}} u', \frac{\sqrt{r^2 + 1}}{r} u'', \mathbf{0} \right).$$

The second fundamental form of  $u$  is

$$II = \langle du, de_{n+1} \rangle = t \left( \frac{r}{\sqrt{r^2 + 1}} du' \cdot du' + \frac{\sqrt{r^2 + 1}}{r} \langle du'', du'' \rangle \right).$$

Denote  $I_k$  as  $k$  order unit matrix,  $\mathbf{0}_k$  as  $k$  order zero matrix. Then we have

$$(h_j^i) = \left( \frac{r}{\sqrt{r^2 + 1}t} I_p \right) \oplus \left( \frac{\sqrt{r^2 + 1}}{rt} I_q \right) \oplus \mathbf{0}_{n-p-q}, \quad H = \frac{pr^2 + q(r^2 + 1)}{nr\sqrt{r^2 + 1}t}, \quad (3.1)$$

$$e^{2\tau} = \frac{p(n-p)r^4 - 2pqr^2(r^2 + 1) + q(n-q)(r^2 + 1)^2}{n-1} \cdot \frac{1}{t^2} := \frac{d}{t^2}. \quad (3.2)$$

A directive calculation tells us that

$$\tau_i = 0, i \neq p + q + 1; \quad \tau_{p+q+1} = -\frac{1}{t}, \quad (3.3)$$

$$\tau_{i,j} = 0, (i, j) \neq (p + q + 1, p + q + 1); \quad \tau_{p+q+1, p+q+1} = \frac{1}{t^2}. \quad (3.4)$$

From (2.8) we get

$$A_j^i = \sum_k e^{-2\tau} I^{ik} A_{kj} = e^{-2\tau} (\tau^i \tau_j - H h_j^i - \tau_j^i - \frac{1}{2} (\sum_k \tau^k \tau_k - H^2) \delta_j^i). \quad (3.5)$$

Let

$$a = \frac{r}{\sqrt{r^2 + 1}}, \quad b = \frac{\sqrt{r^2 + 1}}{r}, \quad c = \frac{pr^2 + q(r^2 + 1)}{nr\sqrt{r^2 + 1}},$$

$$d = \frac{p(n-p)r^4 - 2pqr^2(r^2 + 1) + q(n-q)(r^2 + 1)^2}{n-1}.$$

From (3.1)-(3.5) we know that  $u$  has three constant Blaschke eigenvalues

$$a_1 = \frac{c^2 - 2a - 1}{2d}, \quad a_2 = \frac{c^2 - 2b - 1}{2d}, \quad a_3 = \frac{c^2 - 1}{2d}.$$

Therefore we have the

**Theorem 3.4** Let  $x : \mathbf{M} \rightarrow \mathbb{Q}_1^{n+1}$  be a regular space-like hypersurface in the conformal space  $\mathbb{Q}_1^{n+1}$ . If  $x$  is of parallel conformal second fundamental form  $\mathbb{B}$  and of three constant Blaschke eigenvalues, then it is locally conformally equivalent to the wrapped product embedding

$$u : \mathbb{H}^p(r) \times \mathbb{S}^q(\sqrt{r^2 + 1}) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \subset \mathbb{R}_1^{n+2} \rightarrow \mathbb{R}_1^{n+1},$$

$$(u', u'', t, u''') \mapsto (tu', tu'', u'''),$$

where

$$u' \in \mathbb{H}^p(r), u'' \in \mathbb{S}^q(\sqrt{r^2 + 1}), t > 0, u''' \in \mathbb{R}^{n-p-q-1}, r > 0,$$

for some positive integer  $p, q$ , and  $n - p - q - 1$ .

Summing up the above process, especially using the Theorems 3.1, 3.2, 3.4, we have proved the Theorem B.

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